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MANIFOLDS OF NEGATIVE CURVATURE

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1. Statement of results

1.1. For a Riemannian manifold V we denote by $c^+(V)$ and $c^-(V)$ respectively the upper and the lower bounds of the sectional curvature, by vol (V) the volume, and by $d(V)$ the diameter.

1.2. Let V be an n-dimensional closed Riemannian manifold of negative curvature and $c^-(V) \geq -1$. If $n \geq 8$, then vol $(V) \geq C(1 + d(V))$, where the constant $C > 0$ depends only on n.

Remark. This inequality is exact: For each n there exists an infinite sequence V_i with $d(V_i) \to \infty$, $i \to \infty$, and with uniformly bounded ratio vol $(V_i)/d(V_i)$.

Proof. Take a manifold V of constant negative curvature with infinite group $H_1(V)$ (see [8]) and a sequence of its finite cyclic coverings.

For $n = 4, 5, 6, 7$ we shall prove here the following weaker result: vol $(V) \geq C(1 + d^{1/3}(V))$. Notice that arguments from § 4 show that for $n \geq 4$ an n-dimensional manifold V with $-\varepsilon \geq c^+(V) \geq c^-(V) \geq -1$, $\varepsilon > 0$, satisfies: vol $(V) \geq C(1 + d(V))$ where C depends on n and ε.

1.3. Theorem 1.2 sharpens the Margulis-Heintze theorem (see [6], [4]) stating the inequality vol $(V) \geq C = C_n$. In this paper we prove the following generalization.

1.3A. Let X be a complete simply connected manifold of negative curvature with $c^-(X) > -1$. Let Γ be a discrete group (possibly with torsion) of isometries of V. Then vol $(X/\Gamma) \geq C$, where $C > 0$ depends only on dim (X).

This fact is still true for manifolds of nonpositive curvature with $c^-(X) \geq -1$ and negative Ricci curvature (see [5]). In the homogeneous case this is the Kazhdan-Margulis theorem (see [9]).

The finiteness theorems

1.4. Combining § 1.2 with Cheeger's results (see [1], [4]) we immediately conclude:

For given $n \neq 3$ and $C > 0$ there exist only finitely many pairwise non-diffeomorphic closed n-dimensional manifolds V with $0 > c^+(V) \geq c^-(V) > -1$ and vol $(V) \leq C$.

1.5. Counter-example for $n = 3$. There exists an infinite sequence of 3-di-

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mensional manifolds with uniformly bounded negative curvature and uniformly bounded volume but pairwise not isomorphic one-dimensional homology groups (although with uniformly bounded Betti numbers).

1.6. The homotopy theoretic version of Theorem 1.4 was announced in [7] by Margulis for all n. Although Margulis's statement is incorrect, his geometrical ideas are extremely fruitful and widely used in this paper.

E. Heintze proved in [6] the homotopy type finiteness theorem with diameter instead of volume. In fact the stronger result is true: For given $n = 1, 2, \cdots$ and $C > 0$ there exist only finitely many pairwise non-diffeomorphic closed n-dimenisonal manifolds V with nonpositive curvature and with $c^-(V) > -1$ and $d(V) \leq C$. (For the proof see [5]).

Without assumption $c^+(V) \leq 0$ only the Betti numbers of V can be estimated by curvature and diameter (see [3]).

Pinching

1.7. Another standard application of § 1.2 is the following:

For given n and $C > 0$, there exists an $\varepsilon > 0$ such that under one of the following two conditions a closed n-dimensional manifold V admits a metric of constant negative curvature:

1. $\operatorname{vol}(V) \leq C$, $n \neq 3$, $-1 \geq c^+(V) \geq c^-(V) \geq -1 - \varepsilon$, (for $n = 3$ it is unknown).

2. $|\chi(V)| \leq C$, n is even, $-1 \geq c^+(V) \geq c^-(V) \geq -1 - \varepsilon$, where χ is the Euler characteristic. (Compare with [4]).

Noncompact manifolds

1.8. Let V be a complete Riemannian manifold of negative curvature. If $c^-(V) > -\infty$ and $\operatorname{vol}(V) < \infty$, then V has finite topological type, i.e., V is diffeomorphic to the interior of a compact manifold with boundary.

1.9. D. Kazhdan informed me recently that Margulis proved this fact for manifolds of strictly negative curvature, i.e., with $c^+(V) < 0$. In fact, Theorem 1.8 is still true for real analytic Riemannian manifolds of nonpositive curvature (see [5]), but is not so for C^∞-manifolds (see § 5.1). For the homogeneous case see [9].

2. Groups of isometries

2.1 For an isometry $\gamma: X \to X$ we denote by $\delta_\gamma = \delta_\gamma(x)$, $x \in X$ the displacement $\operatorname{dist}(x, \gamma x)$, and for a group Γ of isometries of X we set $\delta_\Gamma = \delta_\Gamma(x) = \min_{\substack{\gamma \in \Gamma \\ \gamma \neq e}} \delta_\gamma$, where $e \in \Gamma$ is the identity element.

An isometry γ is said to be semisimple if the function δ_γ assumes its minimum on X. If $\min_{x \in X} \delta_\gamma(x) = 0$, then a semisimple isometry is said to be elliptic and hyperbolic otherwise.

2.2. Let X be a complete simply connected manifold of negative curvature. Then for an isometry γ the functions δ_γ and δ_γ^2 are geodesically convex, and δ_γ^2 is strictly convex outside of the set where δ_γ^2 assumes its minimum. (See [2]).

2.3. If X is as above and γ is hyperbolic, then there exists a unique geodesic λ invariant under γ, and δ_γ assumes its minimum on λ. This is obvious and well known.

2.4. We say that a group Γ is almost nilpotent if it possesses a nilpotent subgroup of finite index.

2.5. Let X be as in § 2.2, and Γ an almost nilpotent group of isometries without elliptic elements. Let $\gamma \in \Gamma$ be an hyperbolic isometry, and λ a geodesic invariant under γ. Then λ is invariant under Γ, and Γ is an infinite cyclic group.

Proof. This follows immediately from § 2.3.

2.6. Let X be a complete simply connected manifold of negative curvature, and Γ an almost nilpotent group of isometries. Then there exists a smooth nonegative convex function $\varphi: X \to \boldsymbol{R}$ which is strictly convex at any point $x \in X$ where there is no (non-identical) element from Γ whose displacement assumes its minimum.

Proof. Take a nilpotent subgroup $N \subset \Gamma$ of finite index and any (non-identity) element γ from its center. There are only finitely many isometries $\gamma_1, \cdots, \gamma_k$ conjugate to γ (in Γ). Take $\varphi = \sum_{i=1}^k \delta_{\gamma_i}^2$.

2.7. Corollary. *If X and Γ are as above, then the function δ_Γ does not assumes its maximum; if Γ has no semisimple elements, then δ_Γ has no critical points (in the sense to be explained below).*

2.8. Generally our function $f(x) = \delta_\Gamma(x)$ is not smooth, but near each point $x \in X$ it can be represented as the minimum of smooth functions $f_1, \cdots, f_k$. A point x is said to be noncritical if there exist a tangent vector t at x such that $\langle t, df_i \rangle > 0$, $i = 1, \cdots, k$, and x is said to be critical otherwise.

3. The groups generated by small isometries

3.1. For a group Γ, isometrically acting on X, we denote by $\Gamma_\varepsilon(v)$, $\varepsilon > 0$, $v \in V$, the subgroup generated by all $\gamma \in \Gamma$ with $\delta_\gamma(v) \leq \varepsilon$.

3.2. The Margulis lemma. Let V be a complete Riemannian manifold without closed geodesics of length less than 1 and with $1 \geq c^+(X) \geq c^-(X) \geq -1$. Let Γ be a discrete group of isometries of X. Then there exists a number $\varepsilon = \varepsilon_n > 0$ depending only on $n = \dim X$ such that for any point $x \in X$ the group $\Gamma_\varepsilon(v)$ is almost nilpotent.

For the proof and discussion see [4]. Notice that in [4] this lemma is presented in a different form, but the proof given there serves our present needs as well.

3.3. Proof of Theorem 1.3A. If $\mathrm{vol}\,(X/\Gamma) < \infty$, then the function δ_Γ assumes its maximum, say, at a point $x \in X$. If $\delta_\Gamma(x) \geq \varepsilon = \varepsilon_n$, where ε_n is as above, the proof is finished. If $\delta_\Gamma(x) < \varepsilon$, then the group $\Gamma_\varepsilon(x)$ is almost nilpotent, and the functions δ_Γ and $\delta_{\Gamma_\varepsilon}$ are equal in a neighborhood of x; but this contradicts § 2.7.

3.4. Let X be a complete simply connected n-dimensional manifold of negative curvature with $c^-(X) \geq -1$, and Γ a discrete group of isometries without elliptic elements. Let, $\gamma_1, \cdots, \gamma_i, \cdots \in \Gamma$ be hyperbolic isometries, and let $\Gamma_1, \cdots, \Gamma_i, \cdots \subset \Gamma$ be (uniquely defined) maximal cyclic subgroups containing, $\gamma_1, \cdots, \gamma_i, \cdots$ correspondingly. Denote the sets $(\delta_{\Gamma_i})^{-1}[0, \varepsilon]$ by $A_i \subset X$.

If the groups Γ_i are pairwise not conjugate in Γ, and the number ε is chosen equal to $\varepsilon = \varepsilon_n$ from § 3.2, then for $i \neq j$ and any $\gamma \in \Gamma$, the interection $A_i \cap \gamma A_j$ is empty; if the intersection $A_i \cap \gamma A_i$ is not empty, then $\gamma \in \Gamma_i$.

Proof. Take $x \in A_i$. If $\gamma(x) \in A_j$, $j \neq i$, then the group $\Gamma_\varepsilon(x)$ can not be cyclic because there are $\gamma' \in \Gamma_i$ with $\delta_{\gamma'}(v) \leq \varepsilon$ and $\gamma'' \in \Gamma_j$ with $\delta_{\gamma''}(\gamma(v)) \leq \varepsilon$ and so $\gamma', \gamma^{-1}\gamma''\gamma \in \Gamma_\varepsilon(x)$. On the other hand, it follows from § 3.2 and § 2.5 that $\Gamma_\varepsilon(x)$ is infinite cyclic; so the contradiction proves the first statement and an analogous argument proves the second.

3.5. Corollary. *Let X, Γ and Γ_i be as above. If* vol $(X/\Gamma) < \infty$ *and* $\min_{x \in X} \delta_{\Gamma_i} \leq \frac{1}{2}\varepsilon = \frac{1}{2}\varepsilon_n$, $i = 1, 2, \cdots$ *(ε_n is again from* § 3.2), *then the number of the subgroups Γ_i is finite.*

Proof. The volumes of the sets $B_i = A_i/\Gamma$ are bounded away from zero, the projections $B_i \to X/\Gamma$ are, according to § 3.4, injective and their images do not intersect; therefore the number of B_i and Γ_i is finite.

3.6. Proof of Theorem 1.8. Consider the universal covering $p: X \to V$ with the group $\Gamma = \pi_1(V)$ acting on X. From § 3.5 it follows that there exists a positive number ε' such that for any $x \in X$ the group $\Gamma_{\varepsilon'}(x)$ has no hyperbolic elements, and applying § 3.2 and § 2.7 we conclude that outside of the set $X_0 = (\delta_\Gamma)^{-1}[0, \varepsilon''] \subset X$, $\varepsilon'' = \min(\varepsilon', \varepsilon)$ and $\varepsilon = \varepsilon_n$ from § 3.2 it follows that the function δ_Γ has no critical points. This function is Γ-invariant and so defines a positive function f on V without critical points outside of the set $f^{-1}[0, \varepsilon'']$ $\subset V$, $\varepsilon'' > 0$. Since vol $(V) < \infty$ we have $f(v) \to 0$ as $v \to \infty$, and the application of the Morse theory finishes the proof. (The function $f(v)$ is not smooth, but the Morse theory is obviously applicable for the functions described in § 2.8.)

4. The volume of the tube

4.1. Let λ be a geodesic segment of length l in a manifold X, and let $\theta \in [0, l]$ be the natural parameter in λ. Let $J = J(\theta)$ be a Jacobi field normal to λ with $\langle J(0), J'(0)\rangle = 0$. Set $f(\theta) = \|J(\theta)\|$ and $g(\theta) = \|J'(\theta)\|$. Notice that $f'(\theta) \leq g(\theta)$.

If $0 \geq c^+(X) \geq c^-(X) \geq -1$, $f(0) \leq 1$, $l \geq 1$, then $f(l) \geq f(0) + Cl(\min(g(0), g^3(0)))$, where $C > 0$ depends only on dim X.

Proof. The curvature is nonpositive, so $f' \geq 0$, $f'' \geq 0$ and $(f^2)'' \geq 2g^2$.

Curvature is bounded and so $g' \leq Kf$, where K is the norm of the curvature operator. Using the last inequality we have

$$\begin{aligned}|g(\theta) - g(0)| &\le K\int_0^\theta f(\tau)d\tau \le K\theta f(\theta)\\ &= K\theta\Big(f(0) + \int_0^\theta f'(\tau)d\tau\Big) \le K\theta(f(0) + \theta f'(\theta))\\ &\le K\theta(f(0) + \theta g(\theta))\,,\end{aligned}$$

and for $\theta \le 1$ we have $g(\theta) \ge \dfrac{g(0)}{1+K} - \theta$. Integrating the inequality $(f^2)'' \ge 2g^2$ we obtain for $\theta \le \min\left((1,\ l_0 = \dfrac{g(0)}{2+2K}\right): f^2(\theta) \ge f^2(0) + \dfrac{\theta^2 g^2(0)}{8(1+K)^2}$ and using the convexity of f we have $f(l) \ge f(0) + (l/\theta)(f(\theta) - f(0))$. Combining the last two inequalities and substituting θ by $\min(1, l_0)$ we get the needed estimate.

4.2. Let X be a complet simply connected manifold of negative curvature with $c^-(X) \ge -1$. Consider two points $x_1, x_2 \in X$ with $\operatorname{dist}(x_1, x_2) = m$ and the geodesic μ joining x_1 and x_2. Let t_1 and t_2 be unit tangent vectors at x_1 and x_2 normal to μ, and let α be the angle between t_1 and the vector t_2' at x_1, which is parallel to t_2 along μ. Consider two geodesic segments λ_1, λ_2 of lengths l_1, l_2 starting from x_1, x_2 in the directions t_1, t_2. Denote by y_1, y_2 the second ends of these segments.

If $l_1 \ge 1$, then $\operatorname{dist}(y_1, y_2) \ge m + Cl_1\dfrac{\alpha^3}{1+m^2}$, where $C \ge 0$ depends only on $\dim X$.

This follows from the previous lemma by arguments of the standard comparison type (see [2]).

4.3. Let g be an isometry of the standard unit sphere S^{n-2}. Then it is obvious that for every $N = 1, 2, \cdots$ there exist points $s_1, \cdots, s_N \in S^{n-2}$ with the following property: for any $k = \cdots, -1, 0, 1, \cdots$, $\operatorname{dist}(s_i, g^k s_j) \ge CN^{-1/d}$, where $i \ne j$, $i, j = 1, \cdots, N$, $d = n - 2 - \operatorname{rank}(SO(n-1)) = n - 2 - \operatorname{ent}\left(\dfrac{n-1}{2}\right)$, and $C > 0$ depends only on n.

4.4. Let X be a manifold as in § 4.2, $\mu \subset X$ a geodesic, and let $\gamma: X \to X$ be a hyperbolic isometry keeping μ invariant. Denote by Γ the group generated by γ, and denote by $A_\varepsilon \subset X$, $0 \le \varepsilon \le 1$ the set $\delta_\Gamma^{-1}[0, \varepsilon]$.

Let $n = \dim X$, and let ∂A_ε be the boundary of A_ε. If $\operatorname{dist}(\mu, \partial A_\varepsilon) \ge 2\varepsilon$, and there is a point $y \in A_\varepsilon$ with $l = \operatorname{dist}(y, \mu) \ge 3$, then $\operatorname{vol}(A_{2\varepsilon}/\Gamma) \ge Cl^{P_n}\varepsilon^n$, where $C > 0$ depends on n, $P_n = 1$ for $n \ge 8$, $P_n = \frac{2}{3}$ for $n = 6, 7$, and $P_n = \frac{1}{3}$ for $n = 4, 5$.

Proof. Take the projection $x \in \mu$ from y to μ, and denote by S^{n-2} the sphere of all unit tangent vectors at x normal to μ. The holonomy along μ together with γ defines the isometry g of S^{n-2}. Take points $s_1, \cdots, s_N \in S^{n-2}$ as in § 4.3 with $N = \operatorname{ent}(l^{P_n})$ and the geodesic rays $\lambda_1, \cdots, \lambda_N$ starting at x in directions

$s_1, \cdots, s_N$. Take points $y_i \in \lambda_i \cap \partial A_\varepsilon$, and suppose without loss of generality that $y = y_1$. If for all i dist $(y_i, \mu) \geq \frac{1}{2}l$, then applying § 4.2 and § 4.3 we have dist $(y_i, \gamma^k y_j) \geq \beta$, $i \neq j$, $k = \cdots, -1, 0, 1, \cdots$, and $\beta > 0$ depends only on n. Thus the lemma is proved.

If there is y_i with dist $(y_i, \mu) \leq \frac{1}{2}l$ and $n \geq 3$, then obviously vol $(A_{2\varepsilon}/\Gamma) \geq Cl\varepsilon^n$ which suits us as well.

4.5. Proof of Theorem 1.2. Consider the universal covering $p: X \to V$, and take isometries $\gamma_1, \cdots, \gamma_i, \cdots$ representing the conjugacy classes of isometries corresponding to all simple closed geodesics in X of length $\leq \frac{1}{4}\varepsilon_n$, where ε_n is from § 3.2. Take the sets $A_i = \delta_{\Gamma_i}^{-1}[0, \varepsilon_n]$, where Γ_i is the group generated by γ_i. According to § 3.4 the projections $A_i/\Gamma_i \to V$ are injective, and their images $T_i \subset V$ do not intersect (compare with § 3.5).

Take now two points $v_1, v_2 \in V$ with dist $(v_1, v_2) = d(V)$, and join them by the shortest geodesic segment ν. Consider the union T of all T_i intersecting ν and the ε_n-neighborhood U of ν. It follows from § 4.4 that the set $T \cup U$ provides enough volume to finish the proof.

5. Examples

C^∞-manifolds of monpositive curvature

5.1. Start with a compact C^∞-surface V_i, $i = 1, 2, \cdots$ with the following properties:

a. V_i is diffeomorphic to the torus with two holes.

b. vol $(V_i) \leq 100$.

c. $0 \geq c^+(V_i) \geq c^-(V_i) \geq -1$.

d. Boundary of V_i consists of two geodesics S'_i and S''_{i+2} of lengths $1/2^i$ and $1/2^{i+2}$.

e. Near the boundary each manifold V_i is flat (its curvature is zero).

Denote the product $S_{i+1} \times V_i$ by W_i, where S_{i+1} denotes the circle of length $1/2^{i+1}$. Boundary of W_i consists of two tori $B'_i = S'_i \times S_{i+1}$ and $B''_i = S_{i+1} \times S''_{i+2}$. Each manifold B''_i is canonically isometric to B'_{i+1}, and by identifying all pairs of isometrical tori we obtain the manifold W' with boundary B'_1. The double W of W' is complete C^∞-manifold with finite volume, bounded nonpositive curvature but infinitely generated group $H_1(W)$.

5.2. The previous construction provides many other examples of C^∞-manifolds of nonpositive curvature but without real analytic metrics of nonpositive curvature. The simplest one is the boundary of $V \times V$, where V is a compact surface of positive genus with one hole and with the same geometry at the boundary as manifolds V_i from § 5.1.

Three-dimensional manifolds

5.3. *Horns.* We denote by $\wedge^3$ the 3-dimensional hyperbolic space with curvature -1. Consider a horosphere $S \subset \wedge^3$ and the (convex) horoball B bounded by S.

A horn is, by definition, the quotient $H = B/\Gamma$ where Γ is a discrete group of isometries isomorphic to $Z \oplus Z$. Boundary of H is the flat torus S/Γ.

5.4. Tubes. Consider a geodesic $\mu \subset \wedge^3$ and the set $A(l) \subset \wedge^3$ consisting of all points $a \in \wedge^3$ with dist $(a, \mu) \leq l$. A tube is, by definition, the quotient $B(l) = A(l)/\Gamma$ where Γ is an infinite cyclic group generated by a hyperbolic isometry γ keeping μ invariant. The boundary $\partial B(l)$ of the tube $B(l)$ is isometric to a flat torus, and vol $(B(l)) \leq 100$ vol $(\partial B(l))$.

5.5. Consider a tube $B(l)$ and a horn H, and let $I: \partial H \to \partial B(l)$ be an isometry. Using the normal geodesic coordinates we can canonically extend I to a map $J: U_\varepsilon \to B(l)$ where $U_\varepsilon \subset H$, $\varepsilon < l$, is the ε-neighborhood of ∂H. Denote by $g(J)$ the metric in U_ε induced by J, and denote by g_0 the original metric in U_ε.

It is obvious that if ε is kept fixed and $l \to \infty$, then the metric $g(J)$ C^∞-converges to g_0.

5.6. Let T be a flat torus, and let $h \in H_1(T)$ be an indivisible element. Then there exist a tube $B(l)$ and an isometrical imbedding $I: T \to B(l)$ which maps T isometrically onto the boondary of $B(l)$, and the kernel of the induced homomorphism $I_*: H_1(T) \to H_1(B(l))$ is generated by h. If $h_j \in H_1(T)$ is the sequence of indivisible elements and $h_j \to \infty$, then for the corresponding tubes $B(l_j)$ we also have $l_j \to \infty$.

Proof. Every tube is determined by three parameters: l and two parameters of the isometry γ (shift and rotation), and choosing these parameters in an obvious fashion we construct the needed tubes.

5.7. Take now a complete noncompact orientable 3-dimensional manifold V of curvature -1. One can find in V a compact 3-dimensional submanifold V_0 with the boundary consisting of k flat tori $T_1, \cdots, T_k$ and with the complement $V \setminus \text{Int } V_0$ consisting of k horns bounded by these tori. According to § 5.6 we can find for every T_i a tube with the boundary isometric to T_i, and attaching these tubes to V_0 we obtain a closed manifold with corners at T_i. Moreover Lemma 5.6 shows that by this construction we can obtain infinitely many manifolds with different one-dimensional homologies. On the other hand, using § 5.5 we can smooth the corners providing our closed manifolds with metrics of uniformly bounded negative curvature. That gives the sequence V_i of the manifolds promised in § 1.5. In fact, the above construction gives the manifolds V_i with $-1 \geq c^+(V_i) \geq c^-(V_i) \geq -1 - \varepsilon_i$, where $\varepsilon_i \to 0$, as $i \to \infty$.

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